

ON SUBPROJECTIVITY AND SUPERPROJECTIVITY OF BANACH SPACES

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ABSTRACT. We obtain some results for and further examples of subprojective and superprojective Banach spaces. We also give several conditions providing examples of non-reflexive superprojective spaces; one of these conditions is stable under c_0 -sums and projective tensor products.

1. INTRODUCTION

The classes of subprojective and superprojective Banach spaces were introduced by Whitley [35] to find conditions for the conjugate of an operator to be strictly singular or strictly cosingular. They are relevant in the study of the perturbation classes problem for semi-Fredholm operators [15], which has a positive answer when one of the spaces is subprojective or superprojective [18]. A reflexive space is subprojective (superprojective) if and only if its dual is superprojective (subprojective). In general, however, X being subprojective does not imply that X^* is superprojective, and X^* being subprojective does not imply that X is superprojective, and it is unknown whether the remaining implications are valid [20, Introduction]. Basic examples of subprojective spaces are ℓ_p for $1 \leq p < \infty$ and $L_p(0, 1)$ for $2 \leq p < \infty$ [18, Proposition 2.4]; and $C(K)$ spaces with K a scattered compact are both subprojective and superprojective [18, Propositions 2.4 and 3.4]. Moreover, recent systematic studies of subprojective spaces [28] (see also [13]) and superprojective spaces [20] have widely increased the family of known examples in those classes.

Here we continue the study of subprojective and superprojective Banach spaces. In Section 2 we give some characterisations of these classes of spaces in terms of improjective operators, and apply them to analyse the subprojectivity and superprojectivity of spaces with the Dunford-Pettis property, in particular \mathcal{L}_1 -spaces and \mathcal{L}_∞ -spaces. We show that hereditarily- ℓ_1 spaces with an unconditional basis and hereditarily- c_0 spaces are subprojective, and that $C([0, \lambda], X)$ is subprojective when X is subprojective and λ is an arbitrary ordinal. We also study the subprojectivity and superprojectivity of some \mathcal{L}_∞ -spaces obtained by

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Bourgain and Delbaen [4], which provide counterexamples to some natural conjectures.

In Section 3 we find new examples of non-reflexive superprojective Banach spaces. We show that, if X has property (V) and X^* is hereditarily ℓ_1 , then X is superprojective. In particular, this is true for the spaces in the class that we denote by $Sp(U^{-1} \circ W)$, which includes $C(K)$ spaces with K scattered, the isometric preduals of $\ell_1(\Gamma)$ and Hagler's space JH [21]. Note that JH is a separable space that contains no copies of ℓ_1 and has non-separable dual, hence JH does not admit an unconditional basis. The class $Sp(U^{-1} \circ W)$ is shown to be stable under passing to quotients and under taking projective tensor products and c_0 -sums. We also show that the predual $d(w, 1)_*$ of the Lorentz space $d(w, 1)$ and the Schreier space S are superprojective, although they do not belong to $Sp(U^{-1} \circ W)$, and that their dual spaces are subprojective, but the tensor products $S \hat{\otimes}_\pi S$ and $S \hat{\otimes}_\pi \ell_p$ are not superprojective.

In the sequel, subspaces of a Banach space are assumed to be closed unless otherwise stated. Given a subspace M of a Banach space, J_M and Q_M denote its natural embedding and quotient map. A Banach space X is hereditarily Z if every infinite-dimensional subspace of X contains a subspace isomorphic to Z . Given Banach spaces X and Y , $L(X, Y)$ denotes the set of all (continuous, linear) operators from X into Y , and $K(X, Y)$ denotes the subset of compact operators.

An injection is an isomorphic embedding with infinite-dimensional range, and a surjection is a surjective operator with infinite-dimensional range. A compact space K is said to be scattered, or dispersed, if every nonempty subset of K has an isolated point.

A Banach space X is an $\mathcal{L}_{p,\lambda}$ -space ($1 \leq p \leq \infty$; $1 \leq \lambda < \infty$) if every finite-dimensional subspace F of X is contained in another finite-dimensional subspace E of X whose Banach-Mazur distance to the space $\ell_p^{\dim E}$ is at most λ . The space X is an \mathcal{L}_p -space if it is an $\mathcal{L}_{p,\lambda}$ -space for some λ .

2. SUBPROJECTIVE AND SUPERPROJECTIVE SPACES

We begin by recalling the definitions given in [35] of the concepts we investigate.

Definition. *A Banach space X is called subprojective if every infinite-dimensional subspace of X contains an infinite-dimensional subspace complemented in X , and X is called superprojective if every infinite-codimensional subspace of X is contained in an infinite-codimensional subspace complemented in X .*

The following result [20, Proposition 3.3] is useful to show that some spaces fail subprojectivity or superprojectivity.

Proposition 2.1. *If a Banach space X contains a copy of ℓ_1 , then X is not superprojective and X^* is not subprojective.*

An operator $T: X \rightarrow Y$ is called *strictly singular* if there is no infinite-dimensional subspace M of X such that the restriction TJ_M is an isomorphism. The following, more general concept was introduced by Tarafdar [34].

Definition. *An operator $T: X \rightarrow Y$ is called improjective if there is no infinite-dimensional subspace M of X such that the restriction TJ_M is an isomorphism and $T(M)$ is complemented in Y .*

An operator $T: X \rightarrow Y$ is called *strictly cosingular* if there is no infinite-codimensional subspace N of Y such that $Q_N T$ is surjective. The following characterisation, obtained in [1, Theorem 2.3], shows that strictly cosingular operators are improjective.

Proposition 2.2. *An operator $T: X \rightarrow Y$ is improjective if and only if there is no infinite-codimensional subspace N of Y such that $Q_N T$ is surjective and $T^{-1}(N)$ is complemented in X .*

Next we give some characterisations of subprojectivity and superprojectivity in terms of improjective operators.

Proposition 2.3. *For a Banach space X the following are equivalent:*

- (i) X is subprojective;
- (ii) every improjective operator $T: Z \rightarrow X$ is strictly singular;
- (iii) there exists no improjective injection $J: Z \rightarrow X$.

Proof. (i) \Rightarrow (ii) Suppose that X is subprojective and an operator $T: Z \rightarrow X$ is not strictly singular. Then there exists an infinite-dimensional subspace M of Z such that TJ_M is an isomorphism. Let N be an infinite-dimensional subspace of $T(M)$ complemented in X ; then T is an isomorphism on $M_0 := M \cap T^{-1}(N)$ and $T(M_0) = N$, hence T is not improjective.

(ii) \Rightarrow (iii) It is enough to observe that injections are not strictly singular.

(iii) \Rightarrow (i) Given an infinite-dimensional subspace M of X , the injection $J_M: M \rightarrow X$ is not improjective, so there exists an infinite-dimensional subspace N of M which is complemented in X . Thus X is subprojective. \square

Proposition 2.4. *For a Banach space X the following are equivalent:*

- (i) X is superprojective;
- (ii) every improjective operator $T: X \rightarrow Y$ is strictly cosingular;
- (iii) there exists no improjective surjection $Q: X \rightarrow Y$.

Proof. (i) \Rightarrow (ii) Suppose that X is superprojective and an operator $T: X \rightarrow Y$ is not strictly cosingular. Then there exists an infinite-codimensional subspace N of Y such that $Q_N T$ is surjective. Let M be

an infinite-codimensional subspace complemented in X and containing $T^{-1}(N)$; then $T(M)$ is closed and infinite-codimensional, $Q_{T(M)}T$ is surjective and $T^{-1}T(M) = M$ is complemented in X , hence T is not improjective by Proposition 2.2.

(ii) \Rightarrow (iii) It is enough to observe that surjections are not strictly cosingular.

(iii) \Rightarrow (i) Given an infinite-codimensional subspace N of X , the surjection $Q_N: X \rightarrow X/N$ is not improjective, so there exists an infinite-codimensional subspace M containing N which is complemented in X . Thus X is superprojective. \square

A Banach space X has the Dunford-Pettis property (DPP in short) if every weakly compact operator $T: X \rightarrow Y$ takes weakly convergent sequences to convergent sequences; or, equivalently, if every weakly compact operator $T: X \rightarrow Y$ takes weakly compact sets to relatively compact sets. We refer the reader to [2, Section 5.4] and [22, Section 10] for further information on the DPP. Examples of spaces with the DPP are the \mathcal{L}_∞ -spaces and the \mathcal{L}_1 -spaces [22, Section 10]; in particular, the spaces of continuous functions on a compact $C(K)$ and the spaces of integrable functions $L_1(\mu)$.

The next result establishes some necessary conditions for spaces with the DPP to be subprojective or superprojective.

Proposition 2.5. *Let X be a Banach space satisfying the DPP.*

- (1) *If X is subprojective, then it contains no infinite-dimensional reflexive subspaces.*
- (2) *If X is superprojective, then it admits no infinite-dimensional reflexive quotients.*

Proof. (1) Let R be a reflexive subspace of X . By Proposition 2.3, it is enough to show that the embedding $J_R: R \rightarrow X$ is strictly cosingular, hence improjective, as that would make R finite-dimensional.

Let $Q: X \rightarrow Z$ be an operator such that QJ_R is surjective. Then QJ_R is weakly compact, so Z is reflexive and Q itself is weakly compact, hence completely continuous by the DPP of X . Thus QJ_R is compact, and Z is finite-dimensional.

(2) We could apply Proposition 2.4 to give a proof similar to that of (1), but we choose an alternative one. Take a bounded sequence $(x_n)_{n \in \mathbb{N}}$ in X whose image in the reflexive quotient X/M is weakly convergent but does not have any convergent subsequences. Then Q_M is weakly compact and X has the DPP, so Q_M takes weakly Cauchy sequences to convergent sequences and $(x_n)_{n \in \mathbb{N}}$ cannot have any weakly Cauchy subsequence. Thus X contains a subspace isomorphic to ℓ_1 and it is not superprojective by Proposition 2.1. \square

Corollary 2.6. *A \mathcal{L}_1 -space is subprojective if and only if it contains no infinite-dimensional reflexive subspaces.*

Proof. The direct implication is a consequence of Proposition 2.5. For the converse, observe that each \mathcal{L}_1 -space X is isomorphic to a subspace of some space $L_1(\mu)$ [25]. Therefore, every non-reflexive subspace of X contains a copy of ℓ_1 complemented in X [2, Proposition 5.6.2]. \square

The analogue of Corollary 2.6 for \mathcal{L}_∞ -spaces does not hold. We will see later that there exists a \mathcal{L}_∞ -space Y_{bd} admitting no infinite-dimensional reflexive quotient which is not superprojective.

The next result was essentially proved by Díaz and Fernández [7].

Proposition 2.7. *Every hereditarily- c_0 Banach space is subprojective.*

Proof. It was proved in [7, Theorem 2.2] that if a Banach space X contains no copies of ℓ_1 , then every copy of c_0 in X contains another copy of c_0 which is complemented in X . \square

There are hereditarily- c_0 spaces that admit ℓ_2 as a quotient [14], so they are not superprojective because the corresponding quotient map is improjective (Proposition 2.4).

Proposition 2.8. *Every hereditarily- ℓ_1 Banach space with a (countable or uncountable) unconditional basis is subprojective.*

Proof. It was proved in [12, Theorems 1 and 1a] that every copy of ℓ_1 in a Banach space X with a countable or uncountable unconditional basis contains another copy of ℓ_1 which is complemented in X . \square

Later we will show a hereditarily- ℓ_1 space X_{bd} with a Schauder basis which is not subprojective, so we cannot remove the unconditionality condition in Proposition 2.8.

We already know that $C([0, \lambda], X)$ is subprojective when X is subprojective and $\lambda < \omega_1$ [28]. Next we improve this result.

Theorem 2.9. *Let X be a subprojective Banach space and let λ be an arbitrary ordinal. Then $C([0, \lambda], X)$ is subprojective.*

Proof. Observe that $C([0, \lambda], X) \equiv C_0([0, \lambda], X) \oplus X$, so $C([0, \lambda], X)$ is subprojective if and only if so is $C_0([0, \lambda], X)$ [28, Proposition 2.2]. We will prove that $C_0([0, \lambda], X)$ is subprojective by induction in λ .

Assume that $C_0([0, \mu], X)$ is indeed subprojective for all $\mu < \lambda$. If λ is not a limit ordinal, then $\lambda = \mu + 1$ for some μ and $C_0([0, \lambda], X) \equiv C_0([0, \mu], X) \oplus X$, and the result is clear.

Otherwise, if λ is a limit ordinal, let M be an infinite-dimensional subspace of $C_0([0, \lambda], X)$ and define the projections

$$P_\mu: C_0([0, \lambda], X) \longrightarrow C_0([0, \mu], X)$$

as $P_\mu(f) = f|_{\chi_{[0, \mu]}}$ for each $\mu < \lambda$. If there exists $\mu < \lambda$ such that the restriction $P_\mu|_M$ is not strictly singular, then there exists an infinite-dimensional subspace $N \subseteq M$ such that $P_\mu|_N$ is an isomorphism. Since the range of P_μ is isometric to $C([0, \mu], X)$, which is subprojective by

our induction hypothesis, N contains an infinite-dimensional subspace complemented in $C_0([0, \lambda], X)$ [28, Corollary 2.4].

Assume now, on the other hand, that $P_\mu|_M$ is strictly singular for every $\mu < \lambda$. We will construct a strictly increasing sequence of ordinals $\lambda_1 < \lambda_2 < \dots$ and a sequence $(f_n)_{n \in \mathbb{N}}$ of normalised functions in M such that $\|P_{\lambda_{k-1}}(f_k)\| < 2^{-k}/8$ and $\|P_{\lambda_k}(f_k) - f_k\| < 2^{-k}/8$ for every $k \in \mathbb{N}$, where we write $\lambda_0 = 0$ for convenience. To this end, let $k \in \mathbb{N}$, and assume that λ_{k-1} has already been obtained. By hypothesis, $P_{\lambda_{k-1}}|_M$ is not an isomorphism, so there exists $f_k \in M$ such that $\|f_k\| = 1$ and $\|P_{\lambda_{k-1}}(f_k)\| < 2^{-k}/8$, and then there is $\lambda_k \in (\lambda_{k-1}, \lambda)$ such that $\|P_{\lambda_k}(f_k) - f_k\| < 2^{-k}/8$, which finishes the inductive construction process. Let $F = [f_k : k \in \mathbb{N}] \subseteq M$, which is infinite-dimensional, and define the intervals $I_k = (\lambda_{k-1}, \lambda_k]$ and the operators $T_k = P_{\lambda_k} - P_{\lambda_{k-1}}$ for every $k \in \mathbb{N}$; then $T_k(f) = f\chi_{I_k}$, so each T_k is a norm-one projection and $T_i T_j = 0$ if $i \neq j$.

Let now $g_k = T_k(f_k) = P_{\lambda_k}(f_k) - P_{\lambda_{k-1}}(f_k)$ for each $k \in \mathbb{N}$; then

$$\|g_k - f_k\| \leq \|P_{\lambda_k}(f_k) - f_k\| + \|P_{\lambda_{k-1}}(f_k)\| < 2^{-k}/4,$$

so $1/2 < \|g_k\| < 3/2$ for every $k \in \mathbb{N}$. Note that $C_0([0, \lambda])^* = \ell_1([0, \lambda])$ [11, Theorem 14.24] and $C_0([0, \lambda], X)^* = (C_0([0, \lambda]) \hat{\otimes}_\varepsilon X)^* = C_0([0, \lambda])^* \hat{\otimes}_\pi X^*$ [33, Theorem 5.33], so

$$C_0([0, \lambda], X)^* = \ell_1([0, \lambda]) \hat{\otimes}_\pi X^* = \ell_1([0, \lambda], X^*)$$

and for each $k \in \mathbb{N}$ we can take $x_k \in C_0([0, \lambda], X)^*$ with norm $\|x_k\| < 2$ such that $x_k(g_k) = 1$ and x_k is concentrated on I_k , which makes $(g_n, x_n)_{n \in \mathbb{N}}$ a biorthogonal sequence in $(C_0([0, \lambda], X), C_0([0, \lambda], X)^*)$. In the spirit of the principle of small perturbations [3], let K be the operator defined on $C_0([0, \lambda], X)$ as $K(f) = \sum_{n=1}^\infty x_n(f)(f_n - g_n)$; then

$$\sum_{n=1}^\infty \|x_n\| \|f_n - g_n\| < \sum_{n=1}^\infty 2^{-n}/2 = 1/2,$$

so K is well defined and $U = I + K$ is an isomorphism on X that maps $U(g_k) = f_k$ for every $k \in \mathbb{N}$. Let $G = [g_k : k \in \mathbb{N}]$; then $U(G) = F$ and G is infinite-dimensional.

We will now check that the supremum of the sequence $(\lambda_k)_{k \in \mathbb{N}}$ is λ itself. Assume, to the contrary, that there existed some $\mu < \lambda$ such that $\lambda_k \leq \mu$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, we would have $P_\mu T_k = P_\mu(P_{\lambda_k} - P_{\lambda_{k-1}}) = (P_{\lambda_k} - P_{\lambda_{k-1}}) = T_k$, so $P_\mu(g_k) = g_k$ and P_μ would be an isomorphism on G . But then $P_\mu U^{-1}$ would be an isomorphism on F , where $U^{-1} = I - U^{-1}K$ is a compact perturbation of the identity, so P_μ would be upper semi-Fredholm on $F \subseteq M$, contradicting our assumption that $P_\mu|_M$ is strictly singular.

This means, in turn, that $(x_n(f))_{n \in \mathbb{N}}$ is a null sequence for every $f \in C_0([0, \lambda], X)$, because each x_k is supported on I_k and $\|x_k\| < 2$,

and we can define a projection

$$Q: C_0([0, \lambda], X) \longrightarrow C_0([0, \lambda], X)$$

as $Q(f)(\gamma) = x_k(f)g_k(\gamma)$ if $\gamma \in I_k$, whose range is clearly G . Then G is complemented in $C_0([0, \lambda], X)$, and then so is $U^{-1}(G) = F \subseteq M$, which proves that $C_0([0, \lambda], X)$ is subprojective in this case too. \square

A Banach space X has the Schur property when every weakly convergent sequence in X is convergent. Bourgain and Delbaen [4] obtained two separable \mathcal{L}_∞ -spaces X_{bd} and Y_{bd} which admit Schauder bases and satisfy the following properties:

- X_{bd} has the Schur property, hence it is hereditarily ℓ_1 ; and
- Y_{bd} is hereditarily reflexive and Y_{bd}^* is isomorphic to ℓ_1 .

To study these spaces, we need the following folklore result. We include a proof for the convenience of the reader.

Proposition 2.10. *Every infinite-dimensional separable \mathcal{L}_∞ -space X has a quotient isomorphic to c_0 .*

Proof. Note that X^* contains a sequence $(x_n^*)_{n \in \mathbb{N}}$ equivalent to the unit vector basis of ℓ_1 . Since X is separable, passing to a subsequence we can assume that $(x_n^*)_{n \in \mathbb{N}}$ is weak*-convergent and, subtracting the limit, that $(x_n^*)_{n \in \mathbb{N}}$ is weak*-null.

We consider the operator $T: X \longrightarrow c_0$ defined as $T(x) = (x_n^*(x))_{n \in \mathbb{N}}$. Since its conjugate operator T^* takes the unit vector basis of ℓ_1 to the sequence $(x_n^*)_{n \in \mathbb{N}}$, T^* is an injection, hence T is a surjection. \square

In Proposition 2.10 we can replace “ X separable” by “the unit ball of X^* is weak* sequentially compact” [9, Chapter XIII].

The next result for X_{bd} shows that an analogue of Proposition 2.8 for hereditarily- ℓ_1 spaces is not valid without further hypothesis.

Proposition 2.11. *The spaces X_{bd} and Y_{bd} are neither subprojective nor superprojective.*

Proof. The spaces X_{bd} and Y_{bd} are not subprojective because ℓ_1 or a reflexive space cannot contain an infinite-dimensional \mathcal{L}_∞ -space, and being an \mathcal{L}_∞ -space is inherited by complemented subspaces.

For the other part, Proposition 2.1 implies that X_{bd} is not superprojective, and for Y_{bd} (and also for X_{bd}) we can apply Proposition 2.10 to obtain a surjection $T: Y_{bd} \longrightarrow c_0$. The kernel of T cannot be contained in any infinite-codimensional complemented subspace M , because T would be an isomorphism on the complement of M and Y_{bd} does not contain copies of c_0 . \square

Note that $Y_{bd}^* \simeq \ell_1$ is subprojective, but $X_{bd}^* \simeq C([0, 1])^*$ is not.

3. SUFFICIENT CONDITIONS FOR SUPERPROJECTIVITY

An operator $T: X \longrightarrow Y$ is said to be *unconditionally converging* if there is no subspace M of X isomorphic to c_0 such that the restriction $T|_M$ is an isomorphism. We denote the sets of unconditionally converging and weakly compact operators from X into Y by $U(X, Y)$ and $W(X, Y)$, respectively.

Definition. A Banach space X has property (V) if $U(X, Y) \subseteq W(X, Y)$ for every Banach space Y ; i.e. if every non-weakly compact operator $T: X \longrightarrow Y$ is an isomorphism on a subspace of X isomorphic to c_0 .

It is well known that $C(K)$ spaces have property (V), and it is not difficult to see that property (V) is inherited by quotients. Property (V) relates to superprojectivity because of the following result.

Theorem 3.1. Let X be a Banach space with property (V) such that X^* is hereditarily ℓ_1 . Then X is superprojective.

Proof. Let M be an infinite-codimensional subspace of X . Then $(X/M)^*$ contains a copy of ℓ_1 , so X/M admits an infinite-dimensional separable quotient. Indeed, either X/M has a quotient isomorphic to c_0 or it contains a copy of ℓ_1 [19], in which case it has a quotient isomorphic to ℓ_2 . By passing to that further quotient, we can assume that X/M itself is separable. However, X^* is hereditarily ℓ_1 , so X/M is not reflexive, and the quotient map Q_M is not weakly compact. By property (V), there exists a subspace A of X isomorphic to c_0 such that $Q_M|_A$ is an isomorphism, where $Q_M(A) \simeq c_0$ is complemented because X/M is separable. Then $X/M = Q_M(A) \oplus B$, hence $X = A \oplus Q_M^{-1}(B)$ and $M \subseteq Q_M^{-1}(B)$, so X is superprojective. \square

Remark. In the proof of Theorem 3.1 we need X^* to be hereditarily ℓ_1 to ensure the existence of separable quotients. If this fact can be guaranteed for other reasons (e.g., X separable) we can replace “ X^* hereditarily- ℓ_1 ” by the weaker condition “ X does not admit infinite-dimensional reflexive quotients”.

Following Pietsch [31, 3.2.7], we define $Sp(U^{-1} \circ K)$ as the class of spaces X satisfying that $U(X, Y) \subseteq K(X, Y)$ for every Banach space Y . This class admits a characterisation in terms of property (V) and the Schur property. Let us first state an auxiliary result.

Proposition 3.2. Let X be a Banach space. Then the following are equivalent:

- (i) X^* has the Schur property;
- (ii) X has the DPP and contains no copies of ℓ_1 ;
- (iii) $W(X, Y) \subseteq K(X, Y)$ for every Banach space Y .

Proof. For the equivalence between (i) and (ii), we refer to [8, Theorem 3].

For (iii), assume that X^* has the Schur property, and take $T \in W(X, Y)$; then $T^* \in W(Y^*, X^*) = K(Y^*, X^*)$, hence $T \in K(X, Y)$. Conversely, if there exists a weakly null sequence $(x_n^*)_{n \in \mathbb{N}}$ in X^* that is not norm null, then the operator $T: X \rightarrow c_0$ given by $T(x) = (x_n^*(x))_{n \in \mathbb{N}}$ is weakly compact but not compact. \square

Proposition 3.3. *A Banach space X belongs to $Sp(U^{-1} \circ K)$ if and only if it has property (V) and its dual X^* has the Schur property.*

Proof. Property (V) for X is equivalent to $U(X, Y) \subseteq W(X, Y)$ for every Y , and X^* being Schur is equivalent to $W(X, Y) \subseteq K(X, Y)$ for every Y by Proposition 3.2, which gives the desired result. \square

Corollary 3.4. *Every Banach space in $Sp(U^{-1} \circ K)$ is superprojective.*

Proof. It is enough to observe that spaces with the Schur property are hereditarily ℓ_1 and apply Proposition 3.3 and Theorem 3.1. \square

Corollary 3.5. *A Banach space whose dual is isometric to $\ell_1(\Gamma)$ belongs to $Sp(U^{-1} \circ K)$, hence it is superprojective.*

Proof. The dual $\ell_1(\Gamma)$ has the Schur property, and the space itself has property (V) by [23, Corollary]. \square

Note that, when K is scattered, $C(K)^*$ is isometric to $\ell_1(K)$ [11, Theorem 14.24], and that the space Y_{bd} shows that in the previous Corollary we cannot replace “dual isometric” by “dual isomorphic”.

The next results highlight the interest of the class $Sp(U^{-1} \circ K)$ by showing its stability under quotients, c_0 -sums and projective tensor products.

Proposition 3.6. *The class $Sp(U^{-1} \circ K)$ is stable under passing to quotients.*

Proof. Suppose that X belongs to $Sp(U^{-1} \circ K)$ and $Q: X \rightarrow Z$ is a surjective operator. Given $T \in U(Z, Y)$ we have $TQ \in U(X, Y)$. Then $TQ \in K(X, Y)$, hence $T \in K(Z, Y)$. \square

Proposition 3.7. *Given a sequence $(X_n)_{n \in \mathbb{N}}$ of spaces in $Sp(U^{-1} \circ K)$, the space $c_0(X_n) = \{(x_n)_{n \in \mathbb{N}} : x_n \in X_n, (\|x_n\|)_{n \in \mathbb{N}} \in c_0\}$ belongs to $Sp(U^{-1} \circ K)$.*

Proof. In the case $X_n = X$ for all n , it was proved by Cembranos [5, Teorema 2] that $c_0(X_n)$ has property (V) when each X_n does, and the proof is valid when the spaces X_n are different. Moreover $c_0(X_n)^* \equiv \ell_1(X_n^*)$ has the Schur property when each X_n^* does. \square

Theorem 3.8. *If the spaces X and Y belong to $Sp(U^{-1} \circ K)$, then so does $X \hat{\otimes}_\pi Y$.*

Proof. This is a consequence of two stability results for projective tensor products. Ryan [32, Corollary 3.4] proved that if X^* and Y^* have the Schur property then $(X \hat{\otimes}_\pi Y)^*$ also has the Schur property. Moreover, if X^* is Schur then X contains no copies of ℓ_1 by Proposition 3.2, so any bounded sequence in X must contain a weakly Cauchy subsequence. Since weakly Cauchy sequences in Y^* , which is Schur, must converge, this means that $L(X, Y^*) = K(X, Y^*)$, and it follows from a result of Emmanuele and Hensgen [10, Theorem 2] that if X and Y have property (V) and $L(X, Y^*) = K(X, Y^*)$, then $X \hat{\otimes}_\pi Y$ has property (V). \square

Corollary 3.9. *Let X_1, \dots, X_n be spaces belonging to $Sp(U^{-1} \circ K)$. Then $X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_n$ is superprojective.*

Note that $c_0 \hat{\otimes}_\pi c_0$ is not an \mathcal{L}_∞ -space because $(c_0 \hat{\otimes}_\pi c_0)^{**}$ fails the DPP [16, Corollary 11], and it was proved in [13] that $C(K) \hat{\otimes}_\pi C(L)$ is subprojective when K and L are countable compact.

We do not know if $C(K, X)$ is superprojective when K is scattered and X is superprojective, but the following result gives a partial positive answer. Recall that a Banach space X has *property (u)* when for every weakly Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in X there exists a weakly unconditionally Cauchy series $\sum_{i=1}^\infty y_i$ so that $(x_n - \sum_{i=1}^n y_i)_{n \in \mathbb{N}}$ is weakly null. Banach spaces with an unconditional basis have property (u) [29, Theorem 3].

Proposition 3.10. *Suppose that K is a scattered compact and X is a Banach space with property (u) such that X^* has the Schur property. Then $C(K, X)$ belongs to $Sp(U^{-1} \circ K)$, and so it is superprojective.*

Proof. X contains no copies of ℓ_1 by Proposition 3.2. Since K is scattered and X has property (u), $C(K, X)$ has property (V) [6, Theorem 3]. Moreover $C(K, X)^* \equiv \ell_1(K, X^*)$ has the Schur property, hence $C(K, X)$ belongs to $Sp(U^{-1} \circ K)$ and it is superprojective by Theorem 3.1. \square

Pełczyński proved [30, Proposition 2] that a Banach space with property (u) and containing no copies of ℓ_1 has property (V), so the condition on X in Proposition 3.10 implies $X \in Sp(U^{-1} \circ K)$.

3.1. The Hagler space. In [21], a Banach space JH is constructed such that JH is separable and hereditarily c_0 and JH^* is nonseparable and has the Schur property, hence it is hereditarily ℓ_1 . JH also has property (S), which is defined as follows.

Definition. *A Banach space X has property (S) if every weakly null, non-norm null sequence in X has a subsequence equivalent to the unit vector basis of c_0 .*

Note that JH is subprojective by Proposition 2.7. Also, JH^* is not separable, so JH cannot admit an unconditional basis.

Proposition 3.11. *The space JH belongs to $Sp(U^{-1} \circ K)$, hence JH and $JH \hat{\otimes}_\pi JH$ are superprojective. Moreover JH^* is subprojective.*

Proof. Let us first see that JH belongs to $Sp(U^{-1} \circ K)$. Let $T: JH \rightarrow Y$ be a non-compact operator, and let $(y_n)_{n \in \mathbb{N}}$ be a bounded sequence in JH such that $(T(y_n))_{n \in \mathbb{N}}$ has no convergent subsequence. Since JH contains no copies of ℓ_1 and has property (S), passing to subsequences and taking $u_n := y_{2n} - y_{2n-1}$, we can assume that $(u_n)_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of c_0 and $(T(u_n))_{n \in \mathbb{N}}$ is a seminormalised basic sequence, and then, since $\sum_{n \in \mathbb{N}} T(u_n)$ is weakly unconditionally Cauchy, the sequence $(T(u_n))_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of c_0 [9, Corollary V.7]. Thus $T|_{[u_n]}$ is an isomorphism, hence JH belongs to $Sp(U^{-1} \circ K)$ and Corollary 3.4 implies that JH is superprojective.

To see that JH^* is subprojective, let M be a subspace of JH^* . As JH is separable and JH^* is Schur, we can find a sequence $(x_n^*)_{n \in \mathbb{N}}$ in M equivalent to the unit vector basis of ℓ_1 which is weak*-convergent to some $x_0^* \in X^*$. Let $y_n^* := x_n^* - x_0^*$; by a remark of Johnson and Rosenthal [17, Lemma 3.1.19] we can find a bounded sequence $(y_n)_{n \in \mathbb{N}}$ in X such that $y_k^*(y_l) = \delta_{kl}$. Passing to a subsequence we can assume that $(y_n)_{n \in \mathbb{N}}$ is weakly Cauchy, hence $(y_{2n} - y_{2n-1})_{n \in \mathbb{N}}$ is weakly null. We denote $z_n^* = y_{2n}^*$ and $z_n = y_{2n} - y_{2n-1}$. Since JH has property (S), we can assume that $(z_n)_{n \in \mathbb{N}}$ is equivalent to the unit vector basis of c_0 .

We consider the operators $A: X \rightarrow c_0$ and $B: c_0 \rightarrow X$ defined by $A(x) = (z_n^*(x))_{n \in \mathbb{N}}$ and $Be_n = z_n$. Then $P = BA$ is a projection on X and $R(P^*) \subseteq M + \langle x_0 \rangle$, so M contains a subspace complemented in X^* . \square

The dual JH^* is not superprojective because it contains ℓ_1 .

Proposition 3.12. *Let K be a scattered compact. Then both $C(K, JH) \equiv C(K) \hat{\otimes}_\varepsilon JH$ and $C(K) \hat{\otimes}_\pi JH$ belong to $Sp(U^{-1} \circ K)$, hence they are superprojective.*

Proof. It was proved by Knaust and Odell [24, Theorem 2.1] that property (S) implies property (u). Since JH^* has the Schur property, Proposition 3.10 implies $C(K, JH) \in Sp(U^{-1} \circ K)$.

The result for $C(K) \hat{\otimes}_\pi JH$ follows from Theorem 3.8. \square

3.2. The Schreier space. The Schreier space S is defined as the space of all scalar sequences $x = (x_i)_{i \in \mathbb{N}}$ satisfying

$$\|x\|_S := \sup \left\{ \sum_{i=1}^p |x_{n_i}| : p \leq n_1 < \cdots < n_p \right\} < \infty.$$

It satisfies the following properties:

- (a) The unit vector basis is an unconditional basis for S .
- (b) S is a subspace of $C(\omega^\omega)$; as such, it is hereditarily c_0 .
- (c) S fails the DPP [8, Comments after Theorem 5]. Hence S^* is not Schur (Proposition 3.2) and S does not belong to $Sp(U^{-1} \circ K)$.

Proposition 3.13. *The space S is subprojective and superprojective, and its dual S^* is subprojective but not superprojective.*

Proof. S is subprojective by Proposition 2.7. It is also separable, admits no infinite-dimensional reflexive quotients [27, Theorem B and Corollary 1.10], contains no copies of ℓ_1 , and satisfies property (u) because it has an unconditional basis. Thus S has property (V), and Theorem 3.1 implies that S is superprojective.

Its dual space S^* has an unconditional basis and, since S admits no infinite-dimensional reflexive quotient, S^* contains no reflexive subspace. Thus S^* is hereditarily ℓ_1 , hence it is subprojective by Proposition 2.8 and it is not superprojective by Proposition 2.1. \square

Note that $S \notin Sp(U^{-1} \circ K)$ because S^* is not Schur. This is confirmed by the following result.

Proposition 3.14. *The projective tensor products $S \hat{\otimes}_\pi S$ and $S \hat{\otimes}_\pi \ell_p$ ($1 < p < \infty$) are not superprojective.*

Proof. The dual space of $S \hat{\otimes}_\pi S$ can be identified with $L(S, S^*)$. By [20, Corollary 3.5] it is enough to show that there is a non-compact operator in $L(S, S^*)$.

Given $x = (x_i)_{i \in \mathbb{N}} \in S$, we denote the decreasing rearrangement of $(|x_i|)_{i \in \mathbb{N}}$ by $x^d = (x_i^d)_{i \in \mathbb{N}}$. Note that, for each $n \in \mathbb{N}$, $x_n^d + \dots + x_{2n-1}^d \leq \|x^d\|_S$, so $x_{2n-1}^d \leq \|x^d\|_S/n$ and

$$\|x\|_2^2 = \|x^d\|_2^2 \leq 2(\sum 1/n^2) \|x^d\|_S^2 \leq 2(\sum 1/n^2) \|x\|_S^2,$$

which means that $S \subseteq \ell_2$ and the natural inclusion $J: S \rightarrow \ell_2$ is a bounded operator, and then $J^*J: S \rightarrow S^*$ is not compact.

The proof for $S \hat{\otimes}_\pi \ell_p$ is similar. \square

Observe that the previous argument does not apply to $S \hat{\otimes}_\pi c_0$. We do not know if $S \hat{\otimes}_\pi c_0$ is superprojective.

3.3. The predual of the Lorentz spaces $d(w, 1)$. Given $p \geq 1$ and a non-increasing sequence of positive numbers $w = (w_n)_{n \in \mathbb{N}}$, we consider the space $d(w, p)$ of all sequences of scalars $x = (a_i)_{i \in \mathbb{N}}$ for which

$$\|x\| = \sup \left(\sum_{n=1}^{\infty} |a_{\pi(n)}|^p w_n \right)^{1/p} < \infty,$$

where the supremum is taken over all permutations π of \mathbb{N} . Then $d(w, p)$ endowed with $\|\cdot\|$ is a Banach space [26, Section 3a]. To exclude trivial cases (ℓ_p or ℓ_∞) and normalise the vectors we assume

that $\lim_n w_n = 0$, $\sum_n w_n = \infty$ and $w_1 = 1$. In this case $d(w, p)$ is called a Lorentz sequence space [26, Definition 4.e.1].

The unit vector basis $(e_n)_{n \in \mathbb{N}}$ is a symmetric basis for $d(w, 1)$ and its biorthogonal sequence $(e_n^*)_{n \in \mathbb{N}}$ is a symmetric basis for the predual $d(w, 1)_*$ of $d(w, 1)$. In particular, $d(w, 1)_*$ contains no copies of ℓ_1 .

Proposition 3.15. *The space $d(w, 1)$ is subprojective and its predual $d(w, 1)_*$ is superprojective.*

Proof. The space $d(w, 1)$ is hereditarily ℓ_1 [26, Proposition 4.e.3], hence subprojective by Proposition 2.8.

Since $d(w, 1)_*$ has an unconditional basis, it satisfies property (u) [29, Theorem 3], and $d(w, 1)_*$ does not contain copies of ℓ_1 because $d(w, 1)$ is separable. Then $d(w, 1)_*$ has property (V) and Theorem 3.1 implies that it is superprojective. \square

Proposition 3.16. *The space $d(w, 1)$ fails the Schur property, so $d(w, 1)_* \notin Sp(U^{-1} \circ K)$.*

Proof. Note that $(e_n)_{n \in \mathbb{N}}$ is a symmetric basis in $d(w, 1)$ and

$$\lim_{n \rightarrow \infty} \frac{\|e_1 + \cdots + e_n\|}{n} = \lim_{n \rightarrow \infty} \frac{w_1 + \cdots + w_n}{n} = 0,$$

so $(e_n)_{n \in \mathbb{N}}$ is a normalised weakly null sequence. \square

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